China's 2009–2050 economic growth: A new projection using the Marxian Optimal Growth Model

Abstract

According to Shen (2011) and Onishi (2016), China's economy is predicted to reach zero-growth in 2040 and 2033, respectively. Both studies have attracted significant attention in academia. However, their theoretical models have two inadequacies. First, the Euler equations used to depict the dynamic path of the economy are not derived. Second, the labor growth rate is not incorporated into the model. These inadequacies make their empirical projections arbitrary under strict assumptions. Therefore, we derive the Euler equations of the model using dynamic formulations by incorporating the labor growth rate into our model. Using the Euler equations, we depict the path of the Chinese economy from 2009 to 2050. The result indicates that, in 2026, China's GDP would surpass US GDP. Moreover, around 2050, China's GDP is projected to be almost 2.22 times higher than US GDP, while the GDP capita will be half the US GDP capita.

Keywords: Marxian Optimal Growth Model, Labor growth rate, Euler equation, China, Economic growth path

Introduction

As China's economy has matured, its real GDP has slowed significantly, from 14.2% in 2007 to 6.9% in 2017. The Chinese government has embraced this decline, calling it the "new normal." To analyze this decline, we introduce an optimal growth model call the Marxian Optimal Growth Model; it explains the declining trend of the potential growth rate as an inevitable historical law.

The Marxian Optimal Growth Model was established based on the labor theory of value by Yamashita and Onishi (2002) in order to prove the theory of historical materialism in Friedrich Engels' *Development of Socialism from Utopia to Science*. The model explains capitalist developments under the law of birth, growth, and death from the perspective of the "upper limit" of capital accumulation.Here,the "birth of capitalism" is caused by the industrial revolution,and the "growth " denotes fast-paced capital accumulation. Finally, the "death" will be reflected when the historical role (economic development)will be fully achieved. In this sense, The Marxian Optimal Growth Model is an appropriate model to explain long-term trends of falling economic growth.

The Marxian Optimal Growth Model has attracted significant attention in academia, and has been applied to several empirical studies (Literature on it has been published in Japanese, English, Chinese, and Korean, with studies in *Word Review of Political Economy*, Vol. 2, No. 4 too)[1]. To understand the Marxian Optimal Growth Model, we explain three important considerations of this model according to Yamshita and Onishi (2002), Onishi (2011), and Onishi (2015).

1. The model considers material terms in order to express the fact that accumulation of machinery has been effective for production after the industrial revolution. Specifically, before the industrial revolution, an incremental change in the "means of production" did not result in incremental production capacity, while after the industrial revolution, the incremental change in the "means of production" results in an incremental production capacity[2]. According to Onishi (2011), this kind of relationship can be expressed in terms of an elasticity of production with respect to the means of production, wherein the elasticity has a value of zero in the former case and a positive value in the latter. When expressing this elasticity as a production function, labor input serves as a factor of production in addition to the means of production. This can be expressed only in the form of the Cobb–Douglas function[3], as follows:

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[1] The Marxian Optimal Growth Model has been applied to empirical studies on the Japanese, Chinese, and South Korean economies—Tazoe (2011), Shen (2011), Yin and Yamashita (2013), Onishi (2016), and Li(2018).

[2] The means of production is the "hammer" before industrial revolution, while it is "machinery" after industrial revolution. In this case, giving a second or third hammer to a feudalist craftsperson, who uses the added tool, will not result in any increase in his or her production. However, after industrial revolution, an increase in the number or size of machinery used by a single worker in the modern industry will alone cause an increase in production capacity.

[3] The Cobb–Douglas model has always been an important subject as evidenced by the aggregation problem highlighted by the Cambridge capital controversies of the 1960s. While Shaikh (1974) strictly criticized the aggregate product function, his study showed that the Cobb–Douglas function, with constant "returns to scale," "natural technical change," and "marginal products equal to factor rewards," can be incorporated in empirical studies when the distribution data exhibit constant shares with broad classes of production data. Moreover, Shaikh (2005) showed that the aggregate

$$Y = AK^{\alpha} L^{\beta} \tag{1}$$

($\alpha = 0$ before the industrial revolution; $\alpha > 0$ after the industrial revolution)

Y represents the production of goods for final consumption, A represents "total factor productivity" (a technological coefficient), K is the means of production input, and L is labor input.

2. The Marxian Optimal Growth Model is based on a roundabout production system, as shown in Figure 1. Here the production of machinery is assumed to take place only using labor. This idea is expressed clearly here as it is a model of the labor theory of value. According to Marx's reproduction scheme, there are two sectors in an economic system—the investment goods sector (the means of production K) and the consumption goods sector (the means of consumption Y). Then, the relationship between Y, K, and L can be described as shown in Figure 1. Here, the total labor L is split into two sectors, with s (valued $0 \le s \le 1$) portion of labor diverted to the consumption goods sector (the production of means of consumption), and the portion 1-s, to the investment goods sector (the production of means of production). In this way, the production function of the consumption goods sector is

$$Y = AK^{\alpha} (sL)^{\beta}, \tag{2}$$

And the production function for the investment goods sector can be simplified as

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production function can always be made to work on any data that exhibit roughly constant wage shares, even when the underlying technology is non-neoclassical.

On the other hand, in Yamashita and Onishi (2002), the reason for using the Cobb–Douglas function was explained from the perspective of whether or not the accumulation of capital is effective for production capacity when comparing the production mode after and prior to the industrial revolution. To explain this relationship, the Cobb–Douglas function is the best and only fit of production function.

In the extension model below, we thus take Shaikh (1974) into consideration by setting the Cobb–Douglas function with constant "returns to scale," and "marginal products equal to factor rewards."

$$\dot{K} + \delta K = B(1-s)L. \tag{3}$$

Here K represents the stock of means of production, $\dot{K}$ is the amount of incremental K over a period, B is the labor productivity, and δ is the depreciation rate, with a value of $0 < \delta < 1$.

3. The issue of concern in the Marxian Optimal Growth Model is how to derive the process of long-term capital accumulation. This issue is formularized as an optimization problem. To derive the long-term capital accumulation path, Yamashita and Onishi (2002) formularized this issue as the issue of maximization of production of the means of final consumption using the two production functions introduced above. In this scenario, the Marxian Optimal Growth Model is a normative model that considers the maximization problem under a constrain condition. Here the utility function[4] is

[4] "Utility" is a long standing objection to marginal utility theory by Marxian economists. However, as explained,the Marxian Optimal Growth Model is a normative model. Instead of reflecting the reality of social functioning, the model was first established to explain the necessity of capital accumulation in different stages of society or economic development. Here, whether capital accumulation is necessary or not relates to the definition of capitalism. The conclusion of the Marxian Optimal Growth Model indicates the inevitability of the death of capitalism. To analyze the process of long-term capital accumulation converting to the end of capitalism, Yamashita and Onishi(2002) considered the optimization problem under the constrain condition. Furthermore, here the optimization problem is related to the judgment of a "good society" or "bad society." Thus, we must inevitably introduce the utility function, as it can express this judgment.

On the other hand, In fact, Marx did not object to the use of "utility." In Capital, Volume I, Marx wrote,

"To know what is useful for a dog, one must investigate the nature of dogs. This nature is not itself deducible from the principle of utility. Applying this to man, he that would judge all human acts, movements, relations, etc. according to the principle of utility would first have to deal with human nature in general, and then with human nature as historically modified in each epoch."

inevitably introduced to measure and represent welfare from consuming the production of the means of final consumption. However, considering the diminution of marginal utility per unit of consumption goods, the level of utility to human beings at any moment (instantaneous utility) is Log(Y)[5]. Additionally, we convert the sequence of utility continuing into the future to its present value using the discount rate ρ , which expresses preference between the future and the present. Finally the inter-temporal utility is rewritten as

$$U = \int_0^\infty e^{-\rho t} \log Y(t) dt \tag{4}$$

Here, *e* is the base of the natural logarithm, and (t) appended to *Y* indicates that, in this calculation, *Y* varies over time. *U* is the inter-temporal utility function.

Therefore, the issue is to maximize the inter-temporal utility *U* under the conditions of the two production functions identified—that is,

$$\begin{aligned} &\max U = \int_0^\infty e^{-\rho t} \log Y(t) dt \\ &s.t. \\ &Y(t) = AK(t)^\alpha (s_{(t)} L)^\beta \\ &\dot{K}(t) + \delta K(t) = B(1 - s_{(t)})L \end{aligned} \tag{5}$$

Here, "s.t." is "subject to," which indicates that the equations following "s.t." are the constraint conditions. *Y*(*t*), *K*(*t*), and *s*(*t*) indicate that *Y*, *K*, and *s* vary over time.

Thus, the ratio of total labor power split into two sectors, (s(t):1-s(t)), is set as the instrumental variable for human beings. This is why this model is called as the Marxian Optimal Growth Model—the issue is formularized as an optimization problem in the growth process.

Hence, we can state that, under the assumption of general human nature, (representative agent in Modern Economy), the use of "utility" is acceptable.

In addition, "modern Marxists," including analytical Marxists, especially John Romer, built a strong foundation based on the theory of marginal utility.

Furthermore, this problem has already been explained and defined by Onishi (2011),titled the Marxian Optimal Growth Model, in the *World Review of Political Economy* (Vol. 2, No. 4).

[5] This is because marginal utility diminishes in this form.

> In this sense, the Marxian Optimal Growth Model is a bridge between modern economics and Marxian economics, that is, it specifically explains Marx's theory of historical materialism in the context of modern economics (methods of constrained optimization).

Considering the simplifications of the Marxian Optimal Growth Model, wherein the production goods are assumed to be produced only by the labor force, Kanae (2013) extended this growth model by incorporating capital stock as a factor of production in the production goods sector. We name Kanae's (2013) extended model as the Extension model of the Marxian Optimal Growth Model.

Like the Marxian Optimal Growth Model, the Extension model mainly discusses the inevitability of the falling growth rate and the falling investment ratio of GDP. Consequently, it is considered an appropriate model to analyze the falling trend of economic growth, and accordingly, to forecast the economic growth of countries that have gradually decreasing growth rates. In fact, Shen (2011) and Onishi (2016) adopted this Extension model as a framework to predict China's economic growth. Both studies first calculated the number of times the future optimal capital–labor ratio corresponds to the current situation, and then calculated the necessary period to reach the future optimal situation with the assumption that the current situation is in the optimal path. Shen (2011) predicted 2040 as the year when China's economic growth rate would reach zero, while Onishi (2016) predicted it to be 2033 under the assumption that the total labor force will be constant.

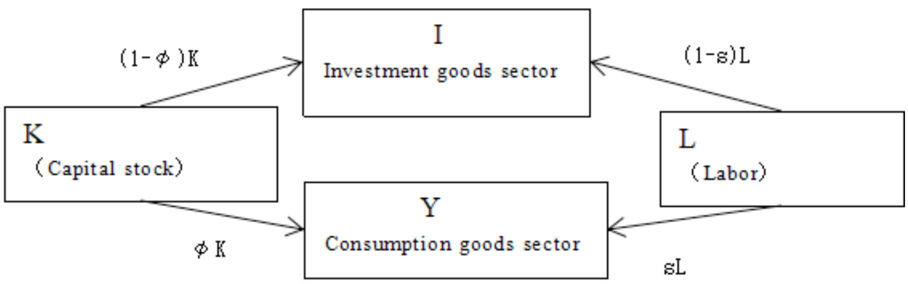
In the process of reaching the zero-growth period, the economic growth rate is projected to change from a high-, to a medium-, and finally, to a low-growth (zero-growth) period. However, Shen (2011) and Onishi (2016) introduced some strong assumptions in the calculation process. For example, both studies calculated the labor and capital shares in the consumption goods sector[6], that

[6] The labor and capital shares—the part of total labor and the part of total capital stock—are used in the consumption goods in the consumption goods sector. We can also simply describe this using the graph below:

is, the capital–labor ratios in both the steady state and the base year. They calculated the optimal labor and capital shares in the consumption goods sector in the steady state at the same time, and both works of research set different assumptions. Onishi (2016) assumed that the labor and capital shares in the production goods sector would decrease linearly in the long term, while Shen (2011) assumed that the capital–labor ratio would continue increasing at the average speed at which the capital ratio increased between 1981 and 2005. Consequently, both studies estimated the time needed to reach zero growth by concurrently using the calculated optimal capital–labor ratio, although such assumptions are unrealistic. Moreover, both studies assumed that the total labor force will be constant. This is very unlikely, as the Chinese industrial/urban workforce is still expanding, even if this expansion has been slowing down.

However, in the theoretical model of the Marxian Optimal Growth Model, the growth path of capital is endogenous. In addition, only if the initial instrumental parameters are guaranteed on saddle paths, can the economy be said to converge toward the steady state point. However, both these points are neglected in the aforementioned studies. In other words, the empirical analyses conducted in both studies contradict the theoretical analyses. Furthermore, the labor growth is not incorporated into the theoretical model. To weaken these assumptions and incorporate the labor growth into the model, that is, to make them more realistic, recalculating the formulation is necessary.

Therefore, we solve the Extension model of the Marxian Optimal Growth Model using dynamic formulations by incorporating the labor growth into the model. We build a complete



In the graph above, s and ϕ are the labor share and capital share, respectively.

The capital is in terms of capital stock. Here, the capacity of the production depends on the holding quantity of the machinery (means of production). The holding quantity of the machinery (means of production) is in terms of the capital stock.

Prediction model without strict assumptions, and name it the Prediction model of the Marxian Optimal Growth Model. Then, using this Prediction model, we depict the path of China's economic growth from 2009 to 2050.

The contributions of this paper can be summarized as follows: (1) The Prediction model established in this paper is an extension of the Marxian Optimal Growth Model. It overcomes the incomplete factors in the former model, and could be used for further empirical research. (2) The current empirical study cannot fully reflect China's economic reality, but would allow adequate policy recommendations.

The remaining paper is structured as follows. In the next section, we explain the basic structure of the Extension model of the Marxian Optimal Growth Model. Then, we introduce the process of establishing the Prediction model in detail. In the third section, we briefly introduce the data used in the model. In the fourth section, we explain the specific stepwise process of the model calculation, which includes describing the method of calculating the data of the initial state, and depicting the growth path of the Chinese economy. Finally, in the last section, we depict the path of China's economic growth from 2009 to 2050, and describe the status of China's economy for 2050.

Basic structure of the Extension model of the Marxian Optimal Growth Model

Kanae (2013) first extended the model adopted by Shen (2011) and Onishi(2015). Like the basic model of the Marxian Optimal Growth Model, the Prediction model incorporates both the production goods and the consumption goods sectors. However, unlike the basic model, both these sectors' processes are observed as a collaboration of labor and capital. The production processes of these sectors are expressed in the form of the Cobb–Douglas function, which is commonly used in modern economics.

Indicating the production goods and consumption goods sectors by the suffixes 1 and 2, respectively, the production function of these two sectors can be introduced as follows:

$$G(K_1, L_1) = A_1 K_1^{\alpha_1} L_1^{\beta_1} \tag{6}$$

$$F(K_2, L_2) = A_2 K_2^{\alpha_2} L_2^{\beta_2} \tag{7}$$

$$\dot{K} = G(K_1, L_1) - \delta K \tag{8}$$

G, F, L, K, and A indicate the production of the production goods sector, production of the consumption goods sector, total labor force, total capital stock, and total factor productivity, respectively. $\dot{K}$ represents the amount of capital stock change with respect to time, and δ ($0 \leq \delta \leq 1$) is the depreciation rate of capital. Furthermore, G, F, and K are time variables, but for convenience, we omit time footnote t.

Like the Marxian Optimal Growth Model, maximize the inter-temporal utility can be represented as:

$$U = \int_0^\infty e^{-\rho t} \log Y dt \tag{9}$$

In the model, the optimal allocation of economic resources is the mission or the ultimate goal of society, pursued over an infinite period. Hence, the following formulas are provided to draw the process of optimization of the entire society over time:

$$\begin{cases} \max\limits_{K_1, K_2, L_1, L_2 \geq 0} \int_0^\infty e^{-\rho t} \log Y dt \\ s.t \\ \dot{K} = G(K_1, L_1) - \delta K \\ Y = F(K_2, L_2) \\ K > K_1 + K_2 \\ L > L_1 + L_2 \end{cases} \tag{10}$$

In the formula above, LogY represents the instantaneous utility at time t, can be written as U_Y[7], and ρ is the rate of time preference. Here, "*s.t.*" means "subject to," which indicates that the equations above are the constraint conditions. As the issue identified here is a conditional maximization problem of inter-temporal utility while satisfying certain conditions, we employ the following Hamiltonian:

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[7] Considering the diminution of marginal utility per unit of consumption goods, we identify the level of utility to a human being *s* at any moment (instantaneous utility) as log(Y).

$$H = LogY + \lambda\{G(K_1, L_1) - \delta K\} + R(K - K_1 - K_2) + w(L - L_1 - L_2) \tag{11}$$

Here, λ is the conjugate variable of *K*, and signifies the price per capital measured by utility. *R* and *w* denote the Lagrangian multiples of *K* and *L*, respectively, and represent the capital price and wage measured by utility[8]. The first-order conditions of this optimizing problem are as follows:

$$\frac{\partial H}{\partial K_1} = \frac{\partial H}{\partial K_2} = 0 \Leftrightarrow \lambda G_K = U_Y F_K = R \tag{12}$$

$$\frac{\partial H}{\partial L_1} = \frac{\partial H}{\partial L_2} = 0 \Leftrightarrow \lambda G_L = U_Y F_L = w \tag{13}$$

$$\frac{\partial H}{\partial K} = \rho\lambda - \dot{\lambda} \Leftrightarrow R - \lambda\delta + \dot{\lambda} = \rho\lambda \tag{14}$$

Furthermore, the optimal capital–labor ratio is obtained as follows:

$$(\frac{K}{L})^* = [A_1(\frac{\alpha_1\delta}{\rho+\delta})^{\alpha_1}(\frac{\alpha_2\beta_1}{\alpha_2\beta_1\delta + \beta_2\{\rho + \delta(1-\alpha_1)\}})^{\beta_1}(\delta L)^{\alpha_1+\beta_1-1}]^{\frac{1}{1-\alpha_1}} \tag{15}$$

At the equilibrium stationary state, the ratios of labor force and capital allocated between the consumption goods sector and production goods sectors are illustrated as follows:

$$K^* : K_1^* : K_2^* = \rho + \delta : \alpha_1\delta : \rho + \delta(1-\alpha_1) \tag{16}$$

$$L : L_1^* : L_2^* = \alpha_2\beta_1\delta + \beta_2\{\rho + \delta(1-\alpha_1)\} : \alpha_2\beta_1\delta : \beta_2\{\rho + \delta(1-\alpha_1)\} \tag{17}$$

After estimating all the parameters of the two production functions—the subjective discount rate ρ and depreciation rate δ—Shen (2011) and Onishi (2016) calculated the optimal capital–labor ratio, the optimal ratio of labor force allocated to the consumption goods sector, and the optimal ratio of capital allocated to the consumption goods sector in the future steady state, using equations (15), (16), and (17). Specifically, both studies calculated the number of times the

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8 Both are defined according to Onishi and Kanae (2015a).These are only definitions to explain the meaning of Lagrangian multiples, which is related to the explanations of the first-order condition equations, (12), (13), and (14). However, it cannot be applied to the empirical study, because these values cannot be measured. For more detailed explanations, please refer to Onishi and Kanae (2015a). Further, this also indicates that it necessary to extend the Marxian Optimal Growth Model to a much completer model that can be used for empirical analysis in another way.

future optimum capital–labor ratio corresponds to the current situation. Then, they calculated the necessary period to reach it. In the process of reaching the zero-growth period, the economic growth rate is predicted to change following a high-, medium-, to low-growth (zero-growth) path. Still, some impractical assumptions exist in Shen (2011) and Onishi's (2016) calculation procedure. That is, the process of the economic growth path capital is exogenous with a strict assumption—Shen (2011) assumed that the trend of the capital–labor ratio's change will maintain the same speed as that from 1981 to 2015. Onishi (2016) assumed that the ratio at which capital and labor are allocated to the production goods sector will decrease linearly. However, such an assumption is unrealistic, and should be revised.

Furthermore, Kanae's (2013) calculation also has a limitation, where he did not provide the growth path itself. Therefore, in order to introduce the economic system's growth path, we prioritize the Euler equations, which indicate these paths directly. Thus, we overcome the limitation of Shen (2011) and Onishi's (2016) projections. Moreover, by assuming the total labor to be constant, these authors neglect its growth in their theoretical and empirical study. In order to yield a realistic result, we thus incorporate the labor growth into our model.

Basic structure of the Prediction model of the Marxian Optimal Growth Model

For the purpose stated in the previous section, we now introduce the process of establishing the Prediction model of the Marxian Optimal Growth Model.

Like the Extension model of the Marxian Optimal Growth Model, the Prediction model incorporates both the production goods and the consumption goods sectors. The goods in both sectors are produced from the collaboration of labor and capital. The production processes of both sectors are expressed in the form of the Cobb–Douglas function. In the Extension model of the Marxian Optimal Growth Model, the total capital (*K*) and total labor (*L*) are allocated in the two sectors following the simple formulation of $L=L_1+L_2$, $K=K_1+K_2$. However, in the Prediction model, capital and labor are allocated between the consumption goods sector and production goods sector with the ratios $\phi(t):(1-\phi(t))$ and $s(t):(1-s(t))$, respectively. Furthermore, both are set as instrumental variables. For convenience, we omitted time footnote t. Then, the amount of labor is equal to the Economically Active Population, and grows at a constant rate *n*, that is,

$$L = L_0 e^{nt} \tag{18}$$

where $L_0 > 0$ is the population for the Economically Active Population in the initial period.

Indicating the production goods and the consumption goods sectors by suffixes 1 and 2, respectively, the production function of these two sectors can be introduced as follows:

$$\begin{cases} I = B[(1-\phi)K]^{\alpha_1}[(1-s)L]^{\beta_1} & (19) \\ Y = A(\phi K)^{\alpha_2}(sL)^{\beta_2} & (20) \\ \dot{K} = I - \delta K & (21) \end{cases}$$

I, *Y*, *L*, *A*, and *B* indicate the production of the production goods sector, production of the consumption goods sector, capital stock, total factor productivity of the consumption goods sector, and total factor productivity of the production goods sector,respectively. α and β indicate the output elasticities of capital and labor, respectively. We assume the function displays constant returns to scale, where the value $\alpha + \beta = 1$. The value δ ($0 \leq \delta \leq 1$) is the depreciation rate. Furthermore, *I*, *Y*, and *K* are time variables, but for convenience, we have omitted time footnote t.

As the production function is assumed to be homogeneous to the degree of 1, we can express it in per capita terms. The equations (18), (19), and (20) can be revised as follows:

$$\begin{aligned} &Y = A(\phi K)^{\alpha_2}(sL)^{\beta_2} \\ &\Rightarrow \frac{Y}{L} = A(\phi K)^{\alpha_2}(s)^{\beta_2}(L)^{-\alpha_2} = A(\frac{K}{L})^{\alpha_2}(s)^{\beta_2} \\ &\Rightarrow y = A(\phi k)^{\alpha_2} s^{\beta_2} \end{aligned} \tag{22}$$

$$\begin{aligned} &I = B[(1-\phi)K]^{\alpha_1}[(1-s)L]^{\beta_1} \\ &\Rightarrow \frac{I}{L} = B[(1-\phi)K]^{\alpha_1}(1-s)^{\beta_1} L^{-\alpha_1} \\ &\Rightarrow i = A[(1-\phi)k]^{\alpha_1}(1-s)^{\beta_1} \end{aligned} \tag{23}$$

$$\begin{aligned} &\dot{K} = I - \delta K \\ &\Rightarrow \dot{k} = i - \delta k - nk \end{aligned} \tag{24}$$

where *i*, *y*, and *k* indicate *Y*, *I*, and *K* in per capita terms.

Like the Extension model of the Marxian Optimal Growth Model, the value of the inter-temporal utility is represented by the next objective function, as follows:

$$\max U = \int_0^{\infty} e^{-\rho t} \log Y dt, \tag{25}$$

which comprises an exponentially discounted instantaneous utility from the consumption goods. Here, ρ refers to the subjective discount rate.

Moreover, we assume that the economy is populated by identical individuals, such that the optimal control problem can be stated in terms of an infinitely lived representative agent with time-invariant utility,

$$Log Y = L \log y = L_0 e^{nt} \log y \tag{26}$$

Here, s and ϕ are instrumental variables that are controlled to maximize the present value of the inter-temporal utility shown in the next objective function:

$$\left\{ \begin{array}{l} \max U = \int_0^{\infty} e^{-(\rho - n)t} \log y dt \\ \\ \quad y = A(\phi k)^{\alpha_2} (s)^{\beta_2} \\ \quad \dot{k} = i - \delta k - nk \\ \text{s.t.} \; i = B[(1-\phi)k]^{\alpha_1} [(1-s)]^{\beta_1} \\ \quad 0 \le s \le 1 \\ \quad 0 \le \phi \le 1 \\ \quad k(0) given \end{array} \right. \tag{27}$$

As the issue identified here is a conditional maximization problem of inter-temporal utility while satisfying certain conditions, we employ the following Hamiltonian:

$$H_c \equiv \log y + \mu \dot{k} \tag{28}$$

$$H_c \equiv \log A + \beta_2 \log s + \alpha_2 \log k + \alpha_2 \log \phi + \mu B(1-s)^{\beta_1} [(1-\phi)k]^{\alpha_1} - \mu \delta k - \mu n k$$

In the above equations, μ is the shadow price of the capital. The first-order conditions of this optimizing problem are as follows:

$$(i)\frac{\partial H_c}{\partial s}=0 \Leftrightarrow \frac{\beta_2}{s}=[(1-\phi)k]^{\alpha_1}\mu B\beta_1(1-s)^{\beta_1-1}$$

$$(ii)\frac{\partial H_c}{\partial \phi}=0 \Leftrightarrow \frac{\alpha_2}{\phi}=\mu B[(1-s)]^{\beta_1}\alpha_1 k^{\alpha_1}(1-\phi)^{\alpha_1-1}$$

$$(iii)\frac{\partial H_c}{\partial k}=\rho\mu-\dot{\mu} \Leftrightarrow \frac{\alpha_2}{k}-\mu\delta-\mu n+\mu B(1-s)^{\beta_1}(1-\phi)^{\alpha_1}\alpha_1(k)^{\alpha_1-1}=\rho\mu-\dot{\mu}$$

$$(iv)\frac{\partial H_c}{\partial \mu}=\dot{k} \Leftrightarrow \dot{k}+\delta k=B[(1-\phi)k]^{\alpha_1}(1-s)^{\beta_1}$$

Additionally, except the four first-order conditions above, the model should satisfy the other optimality condition called the Transversality Condition, which for this model, can be written as follows:

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu k=0 \tag{29}$$

By solving the model, we obtain two Euler equations for this optimal problem.

$$\dot{\phi}=\frac{[\beta_1\delta+\rho+\beta_1 n-\phi\alpha_1 B(\frac{1-s}{1-\phi})^{\beta_1}(k)^{\alpha_1-1}]}{[\frac{\alpha_1\phi-1}{(1-\phi)\phi}+\frac{\beta_1{}^2\alpha_2 s^2}{(1-s)\alpha_1\phi^2\beta_2}]} \tag{30}$$

$$\dot{s}=\frac{[\beta_1\delta+\rho+\beta_1 n-\frac{s\alpha_2 B\beta_1}{\beta_2}(\frac{1-\phi}{1-s})^{\alpha_1}(k)^{\alpha_1-1}]}{[\frac{\beta_1 s-1}{(1-s)s}+\frac{\alpha_1{}^2\beta_2\phi^2}{(1-\phi)\beta_1 s^2\alpha_2}]} \tag{31}$$

The calculation process is too complicated to be introduced briefly, and therefore, we omit this part. The combination of these two formulations with the formulation of capital stock $\dot{k}=B[(1-\phi)k)]^{\alpha_1}[(1-s)]^{\beta_1}-\delta k-nk$ can depict the optimal growth path of the capital–labor ratio, the optimal ratio of the capital allocated in the consumption goods sector ϕ, and the optimal ratio of the labor allocated in the consumption goods sector s.

Furthermore, to confirm the correctness of these two equations, using the two Euler equations and the formula for capital stock, we also calculate the equations of the optimal capital–labor ratio, the optimal ratio of labor force allocated to the consumption goods sector, and the optimal ratio of capital allocated to the consumption goods sector in steady state.

In the steady state, we have the following equations: $\dot{k} = 0, \dot{s} = 0, \dot{\phi} = 0$. The equations of the optimal capital–labor ratio, the optimal ratio of labor force allocated to the consumption goods sector, and the optimal ratio of capital allocated to the consumption goods sector in the steady state are calculated as follows:

$$\begin{aligned}
&\dot{k} = B[(1-\phi)k]^{\alpha_1}(1-s)^{\beta_1} - \delta k - nk \\
&\Rightarrow B[(1-\phi)k)]^{\alpha_1}(1-s)^{\beta_1} - \delta k - nk \\
&\Rightarrow (k)^{\alpha_1 - 1} = \frac{\delta + n}{B[(1-\phi)]^{\alpha_1}(1-s)^{\beta_1}}
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\dot{\phi} = \frac{[\beta_1\delta + \rho + \beta n - \phi\alpha_1 B(\frac{1-s}{1-\phi})^{\beta_1}(k)^{\alpha_1 - 1}]}{[\frac{\alpha_1\phi - 1}{(1-\phi)\phi} + \frac{\beta_1^{\ 2}\alpha_2 s^2}{(1-s)\alpha_1\phi^2\beta_2}]} = 0 \\
&\Rightarrow [\beta_1\delta + \rho + \beta n - \phi\alpha_1 B(\frac{1-s}{1-\phi})^{\beta_1}(k)^{\alpha_1 - 1}] = 0
\end{aligned} \tag{33}$$

If we substitute equation (32) with (33), the share of capital stock for the two sectors can be obtained as follows:

$$\beta_1\delta + \rho + \beta_1 n = \phi\alpha_1 B(\frac{1-s}{1-\phi})^{\beta_1}(\frac{\delta + n}{B(1-\phi)^{\alpha_1}(1-s)^{\beta_1}}) \tag{34}$$

$$\phi^* = \frac{\beta_1\delta + \beta_1 n + \rho}{\delta + \rho + n}. \tag{35}$$

$$1 - \phi^* = 1 - \frac{(1-\alpha_1)\delta + \rho}{\alpha_1\delta + \delta + \rho} = \frac{\beta_1(\delta + n)}{\delta + \rho + n}. \tag{36}$$

This can also be written as:

$$K^* : K_1^* : K_2^* = \delta + \rho + n : \alpha_1(\delta + n) : \beta_1(\delta + n) + \rho \tag{37}$$

Moreover, we obtain:

$$\dot{s} = \frac{[\beta_1\delta + \rho + \beta_1 n - \frac{s\alpha_2 B\beta_1}{\beta_2}(\frac{1-\phi}{1-s})^{\alpha_1}(k)^{\alpha_1 - 1}]}{[\frac{\beta_1 s - 1}{(1-s)s} + \frac{\alpha_1{}^2\beta_2\varphi^2}{(1-\phi)\beta_1 s^2\alpha_2}]} = 0$$

Then, we further obtain:

$$\beta_1\delta + \rho + \beta_1 n - \frac{s\alpha_2 B\beta_1}{\beta_2}(\frac{1-\phi}{1-s})^{\alpha_1}(k)^{\alpha_1 - 1} = 0 \tag{38}$$

If we replace equation (32) with (38), the optimal labor share can be obtained as follows:

$$\beta_1\delta + \rho + \beta_1 n = \frac{s\alpha_2 B\beta_1}{\beta_2}(\frac{1-\phi}{1-s})^{\alpha_1}(\frac{\delta + n}{B(1-\phi)^{\alpha_1}(1-s)^{\beta_1}})$$

$$\Rightarrow s^* = \frac{[\beta_1(\delta + n) + \rho]\beta_2}{[\beta_1(\delta + n) + \rho]\beta_2 + \alpha_2\beta_1(\delta + n)} \tag{39}$$

$$\Rightarrow 1 - s^* = \frac{\alpha_2\beta_1(\delta + n)}{[\beta_1(\delta + n) + \rho]\beta_2 + \alpha_2\beta_1(\delta + n)}. \tag{40}$$

This can also be written as:

$$L^* : L_1^* : L_2^* = \{[\beta_1(\delta + n) + \rho]\beta_2 + \alpha_2\beta_1(\delta + n)\} : \alpha_2\beta_1(\delta + n) : [\beta_1(\delta + n) + \rho]\beta_2 \tag{41}$$

If we substitute both equations (32) and (36) with (281), then the optimal capital–labor ratio can be obtained as follows:

$$(k)^{\alpha_1 - 1} = \frac{\delta + n}{B[\frac{\alpha_1(\delta + n)}{\delta + \rho + n}]^{\alpha_1}\{\frac{\alpha_2\beta_1(\delta + n)}{[\beta_1(\delta + n) + \rho]\beta_2 + \alpha_2\beta_1(\delta + n)}\}^{\beta_1}}$$

$$\Rightarrow (k)^* = \{B[(\frac{\alpha_1}{\delta + \rho + n})]^{\alpha_1}\{(\frac{\alpha_2\beta_1}{[\beta_1(\delta + n) + \rho]\beta_2 + \alpha_2\beta_1(\delta + n)})]^{\beta_1}\}^{\frac{1}{1-\alpha_1}} \tag{42}$$

We observe that equations (37), (41), and (42) reflect Kanae's (2013) results if we assume that the labor growth rate equals to zero, that is, equations (15), (16), and (17). Thus, we have proven that the computation is correct.

The path of China's economic growth from 2009 to 2050

Data gathering and operation

In this study, for the parameters of the two production functions, we refer to Onishi (2016). Compared to Onishi (2016), the methodology of Shen's projection (2011) was not sufficiently refined and thorough, and did not present a theoretical background. Therefore, for the parameters of the two production functions, we refer to Onishi (2016), where these parameters are estimated as follows: $\alpha_1 = 0.911, \alpha_2 = 0.598, \beta_1 = 0.0989, \beta_2 = 0.402$.[9] The total factor productivity of the two sectors is estimated as follows: $A = 0.52564, B = 0.82175$.

The depreciation rate δ is predicted according to the capital accumulation equation $\dot{K} = I - \delta K$. From the database provided by Onishi (2016), we can obtain the annual year's production data and capital stock data from 1980 to 2009 at the 1980 constant price. Then, by statistical regression, we can calculate the value of δ, which is indicated to be approximately 0.141. The subjective discount[10] is indicated to be 0.080065.

In our estimation, the labor growth rate calculation is divided into two parts–population growth rate and the employment participation growth rate. The population growth rate is estimated as follows. First, we use the total population growth rate for China between 2001 and 2050 from the *Word Population Prospects: 2017 Revision* to calculate the annual population growth rates. From

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[9] As the method involves using the inverse matrix, (1-input coefficient matrix)$^{-1}$, which is too complex to calculate, Onishi (2016) adopted a different method. First, the author calculated the ratios of production for investment and consumption in each sector using the Chinese input–output tables for 1982, 1990, 1995, 2000, and 2010. Second, using these ratios, the author divided the industry's capital stock and labor input data between the two sectors of each industry. Here, the labor input data is obtained from the population census data. The capital stock data were obtained from Meng (2012), and then restructured by Onishi (2016) so as to correspond to the input–output tables. In addition, the data for the production of the consumption goods sector and the production goods sectors are sourced from the China Statistical Yearbook, and converted into real term using the 1980 constant price. For more detailed explanations of the date operation, please refer to Onishi (2016).

[10] Here the subjective discount rate was recalculated in the same way as Onishi (2016), but by expanding the database. The subjective discount rate is estimated by applying Piketty's return to capital, (r)>growth rate of the economy (g), theory. With this application, Onishi and Kanae (2015b) also proved the equation $r = g + \rho$ for the Marxian Optimal Growth Model. Furthermore, the value of *r* and *g* was calculated as the average value.

these data, we can calculate the total population from 2001 to 2050. Next, we predict the growth rate of China's employment participation rate in future years. We assume that the change in China's future employment participation rate will maintain the same trend from 2001 to 2017. We thus predict China's future labor force from 2001 to 2050 annually. Finally, using the annual predicted data, we calculate the labor growth rate, which is 0.0576%.

Theoretical background and methodology of the empirical analysis

To depict the path of the Chinese economic growth from 2009 to 2050, the first condition for a path (s,ϕ,k)—with $k>0, 0\le s\le 1, 0\le\phi\le 1$ for all $t\ge 0$—must be the solution to the model and satisfy the system of differential equations, (24), (30), and (31). Under the three given equations, (24), (30), and (31), several possible paths could be solutions. However, only if the path starts at the special point, does it converge toward the steady state point; in this case, the path is called a *saddle-path* in neoclassical economy. Thus, we must calculate s,ϕ corresponding to the given k_0 in initial state. In fact, both Shen (2011) and Onishi (2016) neglected to consider this point. In other words, the empirical analyses conducted in both studies contradict the theoretical analyses.

However, we especially prioritize the paths that start at the historically given k_0, s_0, ϕ_0, which can reflect China's real economic growth path without any technical assumptions, for which we start the path from the actual state in 2009.

After depicting the economic growth path, we compare the maximum values of the capital–labor ratio, the ratio of labor force allocated in the consumption goods sector, and the ratio of capital allocated in the consumption goods sector in the final year with those values in the optimal steady state

First, we calculate the capital–labor ratio, the ratio of labor force allocated in the consumption goods sector, and the ratio of capital allocated in the consumption goods sector in 2009.

Second, we plug the value of $s_{2009},\phi_{2009},k_{2009}$, calculated in the first step, into equations (24), (30), (31). Then, we calculate $\dot{s}_{2009},\dot{\phi}_{2009},\dot{k}_{2009}$. $\dot{s}_{2009},\dot{\phi}_{2009},\dot{k}_{2009}$ refer to the variation from $s_{2009},\phi_{2009},k_{2009}$ to the next period in 2010, for which we simultaneously, we obtain

$s_{2010}, \phi_{2010}, k_{2010}$ and the economic growth rate.

Third, we plug $s_{2010}, \phi_{2010}, k_{2010}$, calculated in the second step, into equations (24), (30) and (31). Then, we calculate $\dot{s}_{2010}, \dot{\phi}_{2010}, \dot{k}_{2010}$. Simultaneously, we obtain $s_{2011}, \phi_{2011}, k_{2011}$ and the economic growth rate and total GDP.

Fourth, we repeat step 2 in the same manner as step 3 until 2050.

Fifth, we depict the growth paths of s, ϕ, k from 2009 to 2050 using all s, ϕ, k obtained above.

Sixth, we calculate the optimal capital–labor ratio, the labor share allocated in the consumption goods sector, and the capital share allocated in the consumption goods sector. Then, we compare these values with those of 2050.

Finally, we calculate the year during which China's GDP would surpass US GDP.

Result analysis

In the preceding subsection, we introduced the methodology and theoretical background that underline our calculation. Specifically, we calculated the optimal capital–labor ratio, optimal labor share, and optimal capital share for the two sectors in steady state. We also calculated the labor and capital shares for the two sectors under a given real value of capital stock (K_{2009}) in 2050. Additionally, we depicted the path of the economic growth from 2009 to 2050.

Using Graph 1, we now compare capital–labor ratio and GDP in different periods.

The capital–labor ratio in 2050 is 116.027 thousand Yuan at the 1980 constant price. It is about 36 times higher than the actual level in 2009 (3.12 thousand Yuan at the 1980 constant price). The total capital stock is $9.307*10^7$ billion Yuan, which is 41 times higher than that in 2009. The total GDP in 2050 will be approximately $1.587*10^7$ billion Yuan at the constant 1980 price, which is about 15 times higher than that in 2009. The total GDP in 2017 was about 2.3 times higher than that in 2009. This means that the GDP in 2050 would be almost 6.52 times higher than that in 2017.

Graph 1. The capital–labor ratio and GDP

	Capital–labor ratio (Thousand Yuan in 1980 constant price)	GDP (Billion Yuan in 1980 constant price)
Actual in 2009	3.120	$1.043*10^{6}$
Projection in 2050	116.027	$1.587*10^{7}$
Optimal value	166.609	$2.199*10^{7}$

Furthermore, using Graph 2, we compare the labor and capital shares between the two aforementioned sectors in different periods. We observe that both capital and labor transform from the production goods sector to the consumption goods sector. We consider this a normal phenomenon of transformation in an economic structure with economic growth.

Graph 2. Resource allocation

			Consumption goods sector	Production goods sector
Actual in 2009	Resource allocation	K	0.2999	0.7001
		L	0.2412	0.7588
Projection in 2050	Resource allocation	K	0.6141	0.3859
		L	0.5485	0.4515
Optimal value	Resource allocation	K	0.4180	0.5820
		L	0.8316	0.1684

However, when comparing the labor and capital shares between the two aforementioned sectors, capital–labor ratio and GDP in different periods, we observe that the predicted values of the capital–labor ratio and GDP in 2050 are lower than the optimal value calculated using the optimal equations. This tendency also reflects the disequilibrium of real economic growth in China, which indicates unbalanced growth between the virtual and the real economy. Conversely, it indicates the need to carry out further supply-side structural reform to lead the development of the real economy. This has been the Chinese government's focus for their economic policy in recent years.

As discussed earlier, the methodology adopted in Shen (2011) and Onishi (2016) to depict China's economic growth path contradicts the theoretical research. Thus, undertaking a projection using the aforementioned methodology once more is necessary. The three growth paths— growth paths of ratio of labor allocated to the consumption goods sector, the ratio of capital allocated to the consumption goods sector, and the total capital, respectively—are described below.

Figure 2 shows the growth path of the labor share allocated to the consumption goods sector. Figure 3 shows the growth path of the capital share allocated to the consumption goods sector. Figure 4 shows the growth of the capital–labor ratio.

Figure 2

Figure 3

Figure 4

Conclusion

This paper incorporates the labor growth rate into the Marxian Optimal Growth Model adopted by Shen (2011) and Onishi (2016), and then derives the Euler equations of the model using dynamic formulations. These extensions allow the Marxian Optimal Growth Model to be a completer Prediction model that can be used in further empirical research without using strict assumptions. We consider this to be a significant theoretical progress.

Using the Prediction model, we depict the China's economic growth path from 2009 to 2050. Furthermore, we compare the maximum values of the capital–labor ratio, the ratio of labor force allocated in the consumption goods sector, and the ratio of capital allocated in the consumption goods sector in the final year with those values in the optimal steady state. The results indicate that the total GDP will be approximately $71.586*10^7$ billion Yuan at the constant 1980 price in 2050, which is about 15 times higher than that in 2009. The total GDP in 2017 was about 2.3 times higher than that in 2009. This means that the predicted GDP in 2050 would be almost 6.52 times higher than that in 2017. In addition, according to the result, we find that, in 2026, China's GDP will grow to $4.74428*10^6$ billion Yuan at the constant 1980 price, which is almost 1.97 times higher than that in 2017. As US GDP is about 1.6 times higher than that of China's, it would be

1.85 times higher than that of China's in 2017[11]. Consequently, China's GDP is projected to surpass US GDP by 2026.

In addition, the result indicates disequilibrium in China's real economic growth. Thus, a growth imbalance exists between the virtual and the real economy. This denotes the need to carry out further supply-side structural reform for real economic development. The Chinese government has already realized this problem, and focused on promoting economic restructuring. We believe that under the powerful and wise leadership of the government, China's economy is bound to progress markedly.

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[11] Here, we assume that the US's GDP will maintain growth at the constant growth rate of 2016, which is 1.46%, from 2016 to 2050.

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